Indian Statistical Institute, Bangalore
B. Math. Second Year
First Semester - Analysis III
Mid-Semester Exam Duration : 3 hours Max Marks 100 Date : Sept 07, 2016

Note: This paper carries a total of 120 marks. Answer as much as you can. The maximum you can score is 100.

1. Let $m \geq 1$ and K a compact subset of $\mathbb{R}^m \times \{0\}$. Then show that there is a function $f : \mathbb{R}^{m+1} \to \mathbb{R}$ such that $(i) f$ is continuous in each of the $m+1$ variables separately, and (ii) f is continuous at $z \in \mathbb{R}^{m+1}$ iff $z \notin K$. [20]

2. (a) State and prove the chain rule for differentiable functions on domains in euclidean space.
   (b) Let $\lambda > 0$, and define $f : \mathbb{R}^m \to \mathbb{R}$ by $f(x) = \|x\|^\lambda$. For what values of λ is f continuously differentiable? Justify your answer. [10 + 10 = 20]

3. (a) Let $x, y, z \ \varepsilon \ \mathbb{R}^m$ and $a < b < c$ be real numbers. Then show that $\frac{z-x}{c-a}$ belongs to the closed line segment joining $\frac{y-x}{b-a}$ and $\frac{z-y}{c-b}$.
   (b) Using part (a), show that if $a, b \in \mathbb{R}^m$, f is a differentiable function on a neighborhood of $[a, b]$ into $\mathbb{R}^n$, then there is an $x \ \varepsilon [a, b]$ such that $\|f(b) - f(a)\| \leq \|b - a\|.\|f'(x)\|$.
   (c) Give an example to show that equality may not be possible in part (b).
   . [10 + 15 + 5 = 30]

4. Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable. Suppose $f'(x)$ is invertible for all $x \in \mathbb{R}^m$. Then show that f is an open map, that is, $f\ (U)$ is open for all open subsets U of $\mathbb{R}^m$. [10]

5. Let $GL(n, \mathbb{R})$ denotes the group of all invertible linear operators on $\mathbb{R}^n$ to $\mathbb{R}^n$. Show that the function $f : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ defined by $f(A) = A^{-1}$ is continuous. [20]

6. Given an example of a function $f : \mathbb{R}^n \to \mathbb{R}$ such that all the partial derivatives of f exist everywhere but f is not differentiable. Justify your answer. [20]